Bank regulation under fire sale externalities∗

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PRELIMINARY and INCOMPLETE

Abstract

This paper examines the optimal design of capital and liquidity regulations under fire sale externalities. The lack of complementary liquidity ratio requirements harms the purpose of capital adequacy requirements by yielding not only inefficiently low risky investment but also more severe financial crises. When capital is regulated but liquidity is not, banks still keep liquid assets for micro-prudential reasons because they can use these resources to protect against liquidity shocks. However, liquidity has also macro-prudential effects: Higher liquidity holdings lead to less severe decreases in asset prices during distress times; but this externality is not internalized by individual banks. Therefore, the liquidity holdings of banks are inefficiently low from a social point of view. Predicting this reaction of banks, the regulator raises the minimum capital ratio requirement to inefficiently high levels, which results in some socially profitable projects also end up being unfunded.

We also show that banks hold less liquid assets compared to the competitive equilibrium when only capital ratio is regulated. There are two ways through which banks can get exposed to fire sale risk: holding excess risky asset and not holding enough liquidity. When the first option is limited by the capital regulation, banks use the second option in order to get closer to their privately optimal level of risk.

Our results indicate that the regulatory framework in the pre-Basel III period, which predominantly focused on capital adequacy requirements, was both inefficient and ineffective in addressing systemic instability caused by fire sales.

∗The analysis and the conclusions set forth are those of the authors and do not indicate concurrence by other members of the research staff or the Board of Governors of the Federal Reserve.
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1 Introduction

The recent financial crisis lead to a redesign of bank regulations with an emphasis on macro-prudential aspects of regulation. In the last two decades capital adequacy requirements have been the dominant tool of bank regulators around the world. Capital requirements were mainly used for two purposes: to enhance the stability of individual financial institutions and to create a level playing field for internationally active banks. The crisis revealed that even financially sound institutions may face liquidity constraints which could undermine financial stability especially when these constraints are faced by many institutions at the same time. Without the unprecedented liquidity and asset price supports of the leading central banks during the crisis, those liquidity problems could have resulted in a dramatic collapse of the financial system. The experience lead to a renewed attention on the regulation of liquidity. A third generation of bank regulation principles, popularly known as Basel III, strengthens the previous Basel capital adequacy accord by adding liquidity requirements.

In this paper we investigate the optimal design of capital and liquidity regulations in a model characterized by systemic externalities generated by asset fire sales. We consider a three period model where a continuum of banks borrow from consumers in the initial period and invest in a long term asset. Banks may face liquidity shocks in the interim period which could result in fire sale of their assets. Banks treat the asset price as given in this market. We start from a competitive equilibrium and ask if a regulator could improve upon this by introducing capital and/or liquidity regulation. In particular, we investigate whether the capital requirements would alone be sufficient to address the systemic externalities or would introduction of liquidity regulation on top of capital adequacy requirements could further improve financial stability and welfare. In order to do this, we compare and contrast two cases: regulation of only capital ratios (partial regulation), and regulation of both capital and liquidity ratios (complete regulation).

We show that the lack of complementary liquidity ratio requirements harms the purpose of capital adequacy requirements by yielding not only inefficiently low risky investment but also more severe financial crises. When capital is regulated but liquidity is not, banks still keep liquid assets for micro-prudential reasons because they can use these resources to protect against liquidity shocks. However, liquidity has also macro-prudential effects: higher liquidity holdings lead to less severe decreases in asset prices in bad times; but this externality not internalized by individual banks. Therefore the liquidity holdings are inefficiently low from a social point of view. Knowing this in advance, the regulator raises capital ratios to inefficiently high levels, which results in some socially profitable projects also go unfunded.

We also show that banks hold less liquid assets compared to the competitive equilibrium when only capital ratio is regulated. There are two ways through which banks can get exposed to fire sale risk: holding excess risky asset and not holding enough liquidity. When the first option is limited by the capital regulation, banks use the second option in order to get closer to their privately optimal
level of risk.

Our results indicate that the regulatory framework in the pre-Basel III period, which predominantly focused on capital adequacy requirements, was both inefficient and ineffective in addressing systemic instability caused by fire sales.

2 Literature Review

Even though capital and liquidity regulations on their own were studied extensively, only a few papers investigated the interaction between these two classical tools of regulators and their optimal determination. Rochet and Vives (2004) show that probability of bank failures, some of which are caused by coordination failures among depositors in their model, can be reduced by ex-ante design of capital or liquidity regulations. Their model implies that capital and liquidity regulations are substitutes from the regulator’s perspective who is trying to achieve financial stability. However, unlike this paper, they do not solve for the optimal determination of these two regulatory tools. They also do not consider how banks would react if the regulator was using only one of these tools.

Acharya et al. (2010) show that simple capital requirements are not always sufficient to address both managerial shirking and asset-substitution (risk-shifting) externalities in banking simultaneously because there is an internal conflict between how the two problems can be addressed: Bank leverage should be high enough to create incentives for creditors to threaten liquidation and deter managerial shocking in monitoring and low enough to induce the bank’s shareholders to avoid excessive risk taking. Therefore, the optimal capital regulation requires a two-tiered capital requirement, with a part of bank capital invested in safe assets. The special capital should be unavailable to creditors upon failure so as to retain market discipline and be available to shareholders only contingent on good performance in order to contain risk-taking. Since the special capital is invested in safe assets, it resembles a liquidity requirement. However, it significantly diverges from reserve requirements, in particular due to the restriction on its distribution to creditors. Therefore, their paper cannot be considered as model on optimal determination of liquidity and capital regulations.

Kashyap et al. (2014) consider an extended version of the Diamond and Dybvig (1983) model to investigate the effectiveness of multiple banking regulations in addressing two common financial system externalities: excessive risk-taking due to limited liability and bank-runs. The central message of the paper is that a single regulation alone, such as capital or liquidity requirements, is never sufficient to correct for the inefficiencies created by these externalities. The authors consider the effectiveness of a combination of capital and liquidity requirements in implementing the social planners solution: Capital requirements can be optimally chosen to eliminate the possibility of a bank-run, while liquidity requirement would reduce the incentives to take excessive risk, by essentially creating tax on the risky investment. However, when the social planner equally cares about the all agents in the economy (who are depositors, bankers and entrepreneurs), such a combination results in lower social welfare compared to the social welfare attained by the use of only capital
requirements. Unlike this paper, their paper does not consider fire sale or pecuniary externalities. This causes a divergence in our results as well. We show that under pecuniary externalities, capital regulations are inefficient unless they are supplemented by liquidity requirements.

Moreover, optimal regulatory mix does not necessarily involve capital or liquidity regulation.

A number of papers, especially after the global financial crisis, drew attention to the macro-prudential role of liquidity requirements, similar to the one considered in this paper. Calomiris et al. (2013) argue that the role of liquidity requirements should be conceived not only as an insurance policy that addresses the liquidity risks in distress times as proposed by Basel III, but also as a prudential regulatory tool to make crisis less likely. However, their paper does not analyze how the liquidity requirements interact with prudential capital regulations.

Even though the literature on the interaction between capital and liquidity requirements is limited, there are studies that examine the interaction between different tools available to regulators. In particular, Acharya (2003) shows that convergence in international capital adequacy standards cannot be effective unless it is accompanied by convergence in other aspects of banking regulation, such as closure policies. Externalities in his model are in the form of cost of investment in the risky asset. He assumes that a bank in one country increases costs of investment for itself and for a bank in the other country as it invests more in the risky asset and thereby creates externalities for the bank in the neighboring country.

This paper is also related to the literature that features financial amplification and asset fire sales which includes the seminal contributions of Fisher (1933), Bernanke and Gertler (1989), Kiyotaki and Moore (1997), Krishnamurthy (2003, 2010), and Brunnermeier and Pedersen (2009). In our model, fire sales result from the combined effect of asset-specificity and correlated shocks that hit an entire industry or economy. This idea, which originated in Williamson (1988) and Shleifer and Vishny (1992), was later employed by fire sale models such as Lorenzoni (2008), Gai et al. (2008), Korinek (2011), and Stein (2012). These latter papers show that under pecuniary externalities that arise from asset fire sales, there exists over-borrowing and hence over-investment in risky assets in a competitive setting compared to the socially optimal solution. We improve upon these papers by adding liquidity to a model of fire sales and consider its optimal regulation together with the regulation of capital ratios.

As opposed to the asset specificity idea discussed above, in Allen and Gale (1994, 1998) and Acharya and Yorulmazer (2008) the reason for fire sales is the limited amount of available cash in the market to buy long-term assets offered for sale by agents who need liquid resources immediately. The scarcity of liquid resources leads to necessary discounts in asset prices, a phenomenon known as “cash-in-the-market pricing.”

The constrained inefficiency of competitive markets in this paper is due to the existence of pecuniary externalities under incomplete markets. The Pareto suboptimality of competitive markets when the markets are incomplete goes back at least to the work of Borch (1962). This idea was
further developed in the seminal papers of Hart (1975), Stiglitz (1982), and Geanakoplos and Polemarchakis (1986) among others. Greenwald and Stiglitz (1986) extended the analysis by showing that pecuniary externalities that by themselves, in general, are not a source of inefficiency, lead to significant welfare consequences when markets are incomplete or there is imperfect information.

In this paper, the incompleteness of markets arises from the financial constraints of bankers in the interim period. In particular, similar to Kiyotaki and Moore (1997) and Korinek (2011), I assume that a commitment problem prevents banks from borrowing the funds necessary for restructuring when liquidity shocks hit. If we complete the markets by allowing banks to borrow from global investors by pledging the future return stream from the assets, there would not be a reason for fire sales and the first best world would be established. In the first best world there would not be a need for either capital regulation nor liquidity regulation because the systemic externality in the financial markets would be eliminated.

3 Model

This model contains three periods, $t = 0, 1, 2$; a continuum of banks and a continuum of consumers each with a unit mass and a financial regulator. There is also a unit mass of global investors. All agents are risk-neutral.

There are two goods in this economy: a consumption good and a capital good (i.e., the liquid and illiquid assets). Consumers are endowed with $e$ units of consumption goods at $t = 0$, and none at later periods.\(^1\)

Banks have a technology that converts consumption goods into capital goods one-to-one at $t = 0$. Capital goods that are managed by a bank until the last period yield $R > 1$ consumption goods per unit. Capital fully depreciates at $t = 2$ and capital goods can never be converted into consumption goods.

Banks choose the level of investment, $n_i$, in the capital good at $t = 0$, and borrow the necessary funds from consumers. We consider deposit contracts that are in the form of simple debt contracts, and assume that the cost of raising deposits is increasing and convex. This assumption could be micro-founded in a liquidity demand model a la Diamond and Dybvig (1983) as shown in Walther (2013). We also assume that banks are protected by limited liability.\(^2\)

Banks also choose how much liquid assets to put aside for each unit of investment in the risky asset. The return on liquid asset is normalized to one. We denote the ratio of liquid assets to risky assets by $b_i$. Therefore, each bank raises a total of $(1 + b_i)n_i$ units of resources from consumers at $t = 0$ at a cost of $D((1 + b_i) n_i)$ which will be paid in the last period. We assume that $D' (\cdot) > 0$

\(^1\)We assume that the initial endowment of consumers is sufficiently large, and it is not a binding constraint in equilibrium.

\(^2\)Limited liability assumption is imposed to match reality and to simplify the analysis of the model. All qualitative results carry on when this assumption is removed.
and $D''(\cdot) > 0$.

All uncertainty is resolved at the beginning of $t = 1$: the economy lands in good times with probability $1 - q$, and in bad times with probability $q$. In good times no banks are hit with shocks, therefore no further actions are taken. Banks keep managing their capital goods and realize the full returns from their investment, $R_{ni}$, in the last period. They make the promised payment, $D((1 + b_i)n_i)$, to consumers, and hence earn a net profit of $R_{ni} + b_i - D((1 + b_i)n_i)$. However, in bad times, the investments of all banks in both countries are distressed. In case of distress, the investment has to be restructured in order to remain productive. Restructuring costs are equal to $c \leq 1$ units of consumption goods per unit of capital. If $c$ is not paid, capital is scrapped (i.e., it fully depreciates).

There are no available domestic resources (i.e., consumption goods) with which to carry out the restructuring of distressed investment at $t = 1$. Only global investors are endowed with liquid resources at this point. Due to a commitment problem, banks cannot borrow the required resources from global investors. Our particular assumption is that individual banks cannot commit to pay their production to global investors in the last period. The only way for banks to raise necessary funds for restructuring is to sell some fraction of the investment to global investors in an exchange of consumption goods.

The capital sales by banks will carry the features of a fire sale: the capital good will be traded below its fundamental value for banks, and the price will decrease as banks try to sell more capital. Banks in each country will retain only a fraction of their assets after fire sales. If the asset price falls below a threshold, the expected return on the assets that can be retained by banks will be lower than the value of the initial investment; hence, banks will become insolvent.

We call this situation a “systemic failure”.

Once it is known that banks are insolvent, the regulator requires the bank owners to manage their capital goods to realize the returns in the last period. The regulator then seizes banks’ returns, and makes the promised payments to depositors. The depositors realize net negative returns in this case. We make sure that the return to deposits is sufficiently high that depositors do at least break even in expected terms at the initial period. If fire sales are sufficiently mild, however, then

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3 For simplicity, we assume that the commitment problem is extreme (i.e., banks cannot commit to pay any fraction of their production to global investors). Assuming a milder but sufficiently strong commitment problem where banks can commit a small fraction of their production, as in Lorenzoni (2008) and Gai et al. (2008), does not change the qualitative results of this paper.

4 An alternative story would be that households come in two generations as in Korinek (2011) and the assets produce a (potentially risky) return in the interim period in addition to the safe return in the final period. In this case, banks can borrow from the first generation households at the initial period because they have sufficient collateral to back their promises in the interim period, but banks cannot borrow from second generation households because the value of all assets are zero in the final period. In this alternative story, second generation households will be the buyers of assets from banks in the two countries and employ them in a less productive technology to produce returns in the final period similar to global investors here.

5 Because all uncertainty is resolved at the beginning of $t = 1$, the expected return to capital retained by banks after fire sales, which is certain at that point, is $R$ units of consumption goods per unit of capital.
banks will have enough assets to make the promised payments to the depositors. In this case banks remain solvent, but compared to good times they make smaller profits. This sequence of events is illustrated in Figure 1.

Banks are subject to regulation in the form of an upper limit on initial investment levels. Regulatory standards are set at the beginning of \( t = 0 \). The regulator determines the maximum investment allowed for banks in its jurisdiction, \( N \), and the minimum liquidity ratio \( B \). Investment levels and liquidity ratios of banks \( i \) have to satisfy \( n_i \leq N \) and \( b_i \leq B \) at \( t = 0 \). The regulatory standards in to maximize the net expected returns on risky investments.

### 3.1 Global Investors

Global investors are endowed with unlimited resources of consumption goods at \( t = 1 \). They can purchase capital, \( y \), from banks in each country at \( t = 1 \) and employ this capital to produce \( F(y) \) units of consumption goods at \( t = 2 \). Let \( P \) denote the market price of the capital good at \( t = 1 \).

Because we have a continuum of global investors, each investor treats the market price as given, and chooses the amount of capital to purchase, \( y \), to maximize net returns from investment at \( t = 2 \).

\[
\max_{y \geq 0} F(y) - Py
\]

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6 This regulation becomes equivalent to a minimum capital ratio requirement when we introduce a costly bank equity capital to the model. We abstract from costly equity capital in the basic model in order to simplify the exposition.

7 We consider the total welfare and we are silent on the distribution of this welfare between banks and consumers. This point is not relevant for our results because all agents are risk neutral and thus have the same utility function.

8 The assumption that there are some global investors with unlimited resources at the interim period when no one else has resources can be justified with reference to the empirical facts during the Asian and Latin American financial crises. Krugman (2000), Aguiar and Gopinath (2005), and Acharya et al. (2011) provide evidence that, when those countries were hit by shocks and their assets were distressed, some outside investors with large liquid resources bought their assets.

9 Price of capital at \( t = 0 \) will be one as long as there is positive investment, and the price of capital at \( t = 2 \) will be zero because capital fully depreciates at this point.
The amount of assets they optimally buy satisfies the following first order conditions $F'(y) = P$. This first order condition determines global investors’ (inverse) demand function for the capital good. Using this, we can define their demand function $Q^d(P)$ as follows:

$$y = F'(P)^{-1} \equiv Q^d(P)$$

We need to impose some structure on the return function of global investors and the model parameters in order to ensure that the equilibrium of this model is well-behaved.

**Assumption 1 (Concavity).**

$$F'(y) > 0 \text{ and } F''(y) < 0 \text{ for all } y \geq 0, \text{ with } F'(0) \leq R.$$  

Assumption Concavity says that although global investors’ return is strictly increasing the amount of assets employed ($F'(y) > 0$), they face decreasing returns to scale in the production of consumption goods ($F''(y) < 0$), as opposed to banks that are endowed with a constant returns to scale technology as described above. $F'(0) \leq R$ implies that global investors are less productive than banks at each level of investment goods employed. The concavity of the return function implies that the demand function of global investors for investment goods is downward sloping (see Figure 2). In other words, global investors will require higher discounts to absorb more assets from distressed banks at $t = 1$. The decreasing returns to scale technology assumption is a reduced way of modeling the existence of industry-specific heterogeneous assets, similar to Kiyotaki and Moore (1997), Lorenzoni (2008), Korinek (2011), and Stein (2012). In this more general setup, global investors would first purchase assets that are easy to manage, but as they purchased more assets, they would need to buy ones that required sophisticated management and operation skills.

The idea that some assets are industry-specific, and hence less productive in the hands of outsiders, has its origins in Williamson (1988) and Shleifer and Vishny (1992).\(^\text{10}\) These studies have claimed that when major players in such industries face correlated liquidity shocks and cannot raise external finance due to debt overhang, agency, or commitment problems, they may have to sell assets to outsiders. Outsiders are willing to pay less than the value in best use for the assets of distressed enterprises because they do not have the specific know-how to manage these assets well and therefore face agency costs of hiring specialists to run these assets.

Empirical and anecdotal evidence suggests the existence of fire sales of physical as well as financial assets.\(^\text{11}\) Fire sales have been shown to exist in international settings as well. For example,\(^\text{10}\)Industry-specific assets can be physical, or they can be portfolios of financial intermediaries because many of these contain exotic tailor-made financial assets (Gai et al., 2008). Examples of industry-specific physical assets include oil rigs and refineries, aircraft, copper mines, pharmaceutical patents, and steel plants.\(^\text{11}\)Using a large sample of commercial aircraft transactions Pulvino (2002) shows that distressed airlines sell aircraft at a 14 percent discount from the average market price. This discount exists when the airline industry is depressed but not when it is booming. Coval and Stafford (2007) show that fire sales exist in equity markets when mutual funds engage in sales of similar stocks.
Krugman (2000), Aguiar and Gopinath (2005), and Acharya et al. (2011) provide significant empirical and anecdotal evidence that during Asian and Latin American crises, foreign acquisitions of troubled countries’ assets were very widely spread across industries and assets were sold at sharp discounts. This evidence suggests that foreign investors took control of domestic enterprises mainly because they had liquid resources whereas the locals did not, even though the locals had superior technology and the know-how to run the domestic enterprises. Further support for this argument comes from the evidence in Acharya et al. (2011): Many foreigners eventually flipped the assets they acquired during the Asian crisis to locals, and usually made enormous profits from such trades.

**Assumption 2** (Elasticity).

\[ \epsilon_{P,y} = -\frac{\partial y}{\partial P} \frac{P}{y} = -\frac{F'(y)}{yF''(y)} > 1 \quad \text{for all } y \geq 0 \]

Assumption *Elasticity* says that global investors’ demand for the capital good is elastic. This assumption implies that the amount spent by global investors on asset purchases, \( P_y = F'(y)y \), is strictly increasing in \( y \). Therefore we can also write Assumption *Elasticity* as \( F'(y) + yF''(y) > 0 \).

If this assumption was violated, multiple levels of asset sales would raise a given amount of liquidity, and multiple equilibria in the asset market at \( t = 1 \) would be possible. This assumption is imposed by Lorenzoni (2008) and Korinek (2011) in order to rule out multiple equilibria under fire sales.\(^{12}\)

Many regular return functions satisfy conditions given by Assumptions *Concavity* and *Elasticity*. Here are two examples that satisfy both assumptions: \( F(y) = R \ln(1+y) \) and \( F(y) = \sqrt{y + (1/2R)^2} \). In our closed form solutions below we will use the first of these examples for its analytical convenience. The following example satisfies Assumption *Concavity*, but not Assumption *Elasticity*: \( F(y) = y(R - 2\alpha y) \) where \( 2\alpha y < R \) for all \( y \geq 0 \).

**Assumption 3** (Range).

\[ 1 + qc < R \leq 1/(1 - q) \]

Assumption *Range* says that the return on investment for banks must not be too low because if they are, equilibrium investment levels will be zero. Nor they must be too high; if they are, equilibrium investment levels will be infinite. In other words, the first inequality says that the expected return to risky project is greater than the return to liquid investment. This assumption, while not crucial for the results, allows us to focus on interesting cases in which equilibrium investment levels are neither zero nor infinite.

\(^{12}\)Gai et al. (2008) provides the leading example where this assumption is not imposed and multiple equilibria in the asset market is therefore considered.
3.2 Competitive Equilibrium

First, we analyze the equilibrium at the interim period in each state of the world, for a given set of investment and liquid assets; then we consider the optimal choice of liquid and illiquid assets at \( t = 0 \). Note that, if good times are realized \( t = 0 \), no further actions need to be taken by any agent. Therefore, at \( t = 1 \) we need only to analyze the equilibrium of the model for bad times.

3.2.1 Crisis and fire sales

Consider the problem of a bank \( i \) if bad times are realized at \( t = 1 \). The bank reaches \( t = 1 \) with a level of investment equal to \( n_i \) and liquid assets of \( b_in_i \) which were chosen at the initial period. The investment is distressed and must be restructured using liquid resources. The investment will not produce any returns in the last period if it is not restructured.\(^{13}\) The bank cannot raise external finance from global investors because it cannot commit to pay them in the last period. Therefore, the only way for the bank to raise the funds necessary for restructuring is to sell some fraction of the investment to global investors and use the proceeds to pay for restructuring costs, whereby it can retain another fraction of the investment.

At the beginning of \( t = 1 \) in bad times, a bank \( i \) decides what fraction of capital to restructure \((\chi_i)\) and what fraction of restructured capital to sell \((1 - \gamma_i)\) to generate the resources for restructuring. Note that \( \gamma_i \) will then represent the fraction of capital that a bank keeps after fire sales.\(^{14}\) Thus the bank takes the price of capital \((P)\) as given, and chooses \( \chi_i \) and \( \gamma_i \) to maximize total returns from that point on

\[
\max_{0 \leq \chi_i, \gamma_i \leq 1} \pi_i = R\gamma_i^s\chi_i^s n_i + P_s(1 - \gamma_i^s)\chi_i^s n_i - c_s\chi_i^s n_i
\]

subject to the budget constraint

\[
P_s(1 - \gamma_i^s)\chi_i^s n_i + n_i b_i - c_s\chi_i^s n_i \geq 0. \tag{4}
\]

The first term in (3) is the (certain) total return that will be obtained from the unsold part of the restructured assets, which are \( \chi_in_i \), in the last period. The second term is the revenue raised by selling a fraction \((1 - \gamma_i)\) of the restructured assets, which are \( \chi_in_i \), at the given market price \( P \). The last term, \( c\chi_i n_i \), gives the total cost of restructuring. Budget constraint (4) says that the sum of the liquid assets carried from the initial period and the revenues raised by selling capital must be

\(^{13}\)For example, if the assets are physical, restructuring costs can be maintenance costs or working-capital needs.

\(^{14}\)Following Lorenzoni (2008) and Gai et al. (2008), I assume that banks have to restructure an asset before selling it. Basically, this means that bank receive the asset price \( P \) from global investors, use a part, \( c \), to restructure the asset, and then deliver the restructured assets to global investors. Therefore banks will sell assets only if \( P \) is greater than the restructuring cost, \( c \). We could assume, without qualitatively changing our results, that it is the responsibility of global investors to restructure the assets that they purchase. However, the model is more easily solved using the current story.
greater than or equal to the restructuring costs.

By Assumption \textit{Concavity}, the equilibrium price of capital must satisfy \( P \leq F'(0) \leq R \), otherwise global investors will not purchase any capital. Later on, we will show that in equilibrium we must also have \( c < P \). For the moment, we will assume that the equilibrium price of assets satisfies

\[
eq P < R
\]  

(5)

Now, consider the first order conditions of the maximization problem (3) while ignoring the constraints

\[
\frac{\partial \pi_i}{\partial \chi_i} = [R\gamma_i^s + P_s(1 - \gamma_i^s) - c_s]n_i
\]  

(6)

\[
\frac{\partial \pi_i}{\partial \gamma_i} = (R - P_s)\chi_i^s n_i
\]  

(7)

From (7) it is obvious that \( \pi_i \) is increasing in \( \gamma_i \) because \( P \leq R \) by (5): when the price of capital goods is lower than the return that banks can generate by keeping them, banks want to retain a maximum amount. Choosing \( \gamma_i \) as high as possible implies that the budget constraint will bind. Hence, from (4) we obtain that the fraction of capital goods retained by banks after fire sales is

\[
\gamma_i^s = 1 + \frac{b_i - c_s}{P_s}
\]  

(8)

The fraction banks retain after fire sales (\( \gamma_i \)) is increasing in the price of the capital good (\( P \)) and the liquidity ratio (\( b_i \)) and decreasing in the cost of restructuring (\( c \)). From (8) we can also obtain the total capital supply of a bank \( i \) as

\[
Q_i^s(P, n_i, b_i) = (1 - \gamma_i)n_i = \frac{c - b_i}{P_s} n_i
\]  

(9)

for \( c < P \leq R \). This supply curve is downward-sloping and convex, which is standard in the fire sales literature. A negative slope implies that if there is a decrease in the price of assets banks have to sell more assets in order to generate the resources needed for restructuring. This is because banks are selling a valuable investment at a price below the fair value for them due to an exogenous pressure (e.g., paying for restructuring costs).

On the other hand, using (8) we can write the first order condition (6) as

\[
\frac{\partial \pi_i}{\partial \chi_i^s} = R\gamma_i^s n_i \geq 0
\]  

(10)

Equation (10) shows that revenues are increasing in \( \chi_i \) at \( t = 1 \). Therefore, banks will optimally choose to restructure the full fraction of the investment (\( \chi_i = 1 \)). In other words, scrapping of capital will never arise in equilibrium.
Figure 2: Equilibrium in the Capital Goods Market and Comparative Statics

Note that if the capital price is greater than $R$, banks want to sell all the capital goods they have because they can get at most $R$ per unit by keeping and managing them. If the price is lower than $c$, however, they will optimally scrap all of their capital ($\chi_i = 0$). As discussed above, prices above $R$ and below $c$ will never arise in equilibrium. The total asset supply curve of banks is plotted in Figure 2 for an initial total investment in the two countries of $\tilde{N}$.

3.2.2 Equilibrium in the Capital Market at t=1

Equilibrium price of capital goods, $P^*$, will be determined by the market clearing condition

$$E(P^*, n) = Q^d(P^*) - Q^s(P^*, n, b) = 0 \quad (11)$$

The condition above says that the excess demand in the capital market, denoted by $E(P, n_i, b_i)$, is equal to zero at the equilibrium price. $D(P)$ in Equation (11) is the demand function of global investors which was obtained from the first order conditions of global investors’ problem as shown by (2). $Q^s(P, n, b)$ is the total supply of capital goods obtained by integrating (9) over $i$.

This equilibrium is illustrated in the left panel of Figure 2. Note that the equilibrium price of capital at $t = 1$ will be a function of the total initial investment in the risky asset and safe assets. Therefore, from the perspective of the initial period we denote the equilibrium price as $P^*(n_i, b_i)$.

How does a change in the initial risky investment level affect the price of capital at $t = 1$? Lemma 1 shows that if investment into the risky asset increases at $t = 0$, a lower price for capital will be realized in the fire sales state at $t = 1$.

**Lemma 1.** $P^*(n, b)$ is decreasing in $n$.

Lemma 1 implies that higher investment in the risky asset increases the severity of the financial
crisis by lowering the asset prices. This effect is illustrated in the right panel of Figure 2. Suppose that initial risky investment level increases. In this case, banks will have to sell more assets at each price, as can be seen from individual supply function given by (9). Graphically, the total supply curve will shift to the right, as shown by the dotted-line supply curve in the right panel of Figure 2, which will cause a decrease in the equilibrium price of capital goods. Lower asset prices, by contrast, will induce more fire sales by banks due to the downward-sloping supply curve. This additional result is formalized in Lemma 2.

**Lemma 2.** *Equilibrium fraction of assets sold, \( 1 - \gamma^*(n, b) \), is increasing in \( n \).*

Together lemmas 1 and 2 imply that a higher initial investment in the risky investment by one bank creates negative externalities for other banks by making financial crises more severe (i.e., via lower asset prices according to Lemma 1) and more costly (i.e., more fire sales according to Lemma 2).

## 4 Competitive Equilibrium

Each bank \( i \) at \( t = 0 \) chooses the level of investment in risky asset \( n_i \) and liquidity holdings, as a ratio of investment in risky asset \( b_i \), to maximize expected profits given by

\[
\max_{n_i, b_i} \Pi_i(n_i, b_i) = (1-q)\{R+b_i\}n_i + q\{I(b_i < c)R_{\gamma_i} + I(b_i \geq c)\{R+b_i - c\}\}n_i - D(n_i(1+b_i)) \tag{12}
\]

subject to the budget constraint at \( t = 1 \)

\[
P(1 - \gamma_i)n_i + b_i n_i - c n_i \geq 0
\]

Here \( b_i n_i \) is the total liquidity holdings of bank \( i \). We will interpret \( b_i \) as the liquidity ratio of bank \( i \).

Let us now check whether the budget constraint at \( t = 1 \) is going to bind. Whether the constraint is going to bind depends on banks’ behavior as well as exogenous shocks. The budget constraint does not bind if there is no additional liquidity requirement, i.e. liquidity shock, at \( t = 1 \). In case of a liquidity shock, the constraint binds depends on how much liquidity holdings a bank has carried to \( t = 1 \). If a bank has chosen \( b_i < c \), the constraint will be binding and non-binding if \( b_i \geq c \).

Below we formally show that the optimal behavior of banks requires that the constraint will bind.

Assume \( b_i = c \). [Also assume that we have shown \( b_i > c \) is never optimal.] Corresponding first order conditions with respect to \( n_i \) and \( b_i \) are respectively:

\[
(1-q)(R+b_i) + qR = D'(n_i(1+b_i))(1+b_i)
\]
\[(1-q)n_i + qn_i = D'(n_i(1+b_i))n_i = D'(n_i(1+b_i)) = 1\]

Combining the two equations and plugging \(b_i = c\) implies \(R+(1-q)b_i = 1+b_i \implies R+(1-q)c = 1+c\), which contradicts with the assumption \(R-cq > 1\).

Assume \(b_i < c\). Later we will show that this condition is satisfied. Corresponding first order conditions with respect to \(n_i\) and \(b_i\) are respectively:

\[(1-q)(R+b_i) + qR\gamma_i = D'(n_i(1+b_i))(1+b_i) \text{ where } \gamma_i = 1 + \frac{b_i - c}{P} \tag{13}\]

\[(1-q)n_i + qR\frac{1}{P}n_i = D'(n_i(1+b_i))n_i \tag{14}\]

Combining the two equations

\[(1-q)R + (1-q)b_i + qR + qR\left(\frac{b_i - c}{P}\right) = (1-q) + (1-q)b_i + \frac{qR}{P} + \frac{qR}{P}b_i\]

Solving for \(P\) gives

\[P = \frac{qR(1+c)}{R-1+q}\]

\(P\) is increasing in \(c\) and \(R\), and decreasing in \(q\) up to a point, until \(\bar{q} = 2-R\), then increasing.

**Lemma 3.** \(R \geq P\).

**Proof.**

\[R \geq P \implies R \geq \frac{qR(1+c)}{R-1+q} \implies 1 \geq \frac{q(1+c)}{R-1+q} \implies R - 1 + q \geq q(1+c)\]

which is guaranteed to hold by the assumption \(R-cq > 1\).

**Lemma 4.** \(P > c\)

**Proof.**

\[P > c \implies qR + qRc > Rc - c + qc \implies c - cq > R(c - q - qc)\]

Replacing \(R\) with \(1-cq\) due to the assumption \(R-cq > 1\),

\[c-cq > R(c - q - qc) > (cq + 1)(c - q - qc) = c^2q - c^2q^2 - q^2q^2 + c - q - qc\]

implies \(0 > c^2q - c^2q^2 - q^2q^2 - q\), which must hold given that \(c < 1\) and \(q < 1\).

**Functional Assumptions:**
Demand side: Suppose that the return function of global investors is given by $F(y) = R \ln(1+y)$. For this return function we obtain the (inverse) demand function as

$$P = F'(y) = \frac{R}{1+y} \quad \text{and hence} \quad y = F'^{-1}(P) = \frac{R - P}{P} \equiv Q^d(P) \quad (15)$$

Moreover suppose that the cost of deposits is given by $D(x) = x + dx^2$, and hence $D'(\cdot)$ is increasing, i.e. $D'(x) = 1 + 2dx$

Plug $P$ into demand side and define $\tau$ as follows

$$y = \frac{R}{P} - 1 = \frac{R - 1 + q}{q(1+c)} - 1 = \tau \quad (16)$$

We defined $\tau$ here in terms of exogenous variables. [It can also be written as $\frac{R}{P} = \tau + 1$, and we will use this below.] Equating this to supply side, $(1-\gamma)n = \tau$. Due to symmetric and measure one banks, in equilibrium $b_i = b$ and $n_i = n$.

$$(c - b)n = P\tau \implies n = \frac{P\tau}{c-b}$$

Given $b$, this equation solves for $n$.

Plugging $\frac{R}{P} = \tau + 1$ and $D'(n(1+b)) = 1 + 2dn(1+b)$ into equation (14)

$$1 - q + q(\tau + 1) = D'(n(1+b)) = 1 + 2dn(1+b) \implies 1 - q + q\tau + q = 1 + q\tau = 1 + 2d\frac{P\tau}{c-b}(1+b)$$

where we also used $n = \frac{P\tau}{c-b}$.

By plugging $P = \frac{R}{\tau+1}$ and simplifying we obtain the liquidity ratio in the competitive equilibrium

$$cq\tau - 2d\frac{\tau R}{\tau + 1} = b(2d\frac{R}{\tau+1} + q)\tau \text{ solves for } b$$

$$b = \frac{cq\tau - 2d\frac{\tau R}{\tau+1}}{(2d\frac{R}{\tau+1} + q)\tau} = \frac{cq - 2d\frac{R}{\tau+1}}{2d\frac{R}{\tau+1} + q}$$

**Proposition 1.** The liquidity ratio in the competitive equilibrium ($b$) is increasing in the size of the liquidity shock ($c$), the return to the risky asset ($R$) and the probability of the bad state ($q$), and decreasing in the marginal cost of funds ($d$).
Proof.

\[
\frac{\partial b}{\partial c} = \frac{\{q - 2dR(\tau + 1)^{-2}(1)\frac{\partial \tau}{\partial c}\}[2d\frac{R}{\tau + 1} + q] - [cq - 2d\frac{R}{\tau + 1}]2dR(\tau + 1)^{-2}(1)\frac{\partial \tau}{\partial c}}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{2dR(\tau + 1)^{-2}(1)\frac{\partial \tau}{\partial c}[-2d\frac{R}{\tau + 1} - q - cq + 2d\frac{R}{\tau + 1}] + q[2d\frac{R}{\tau + 1} + q]}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{2dR(\tau + 1)^{-2}(1)\frac{\partial \tau}{\partial c}(-q)(1 + c) + q[2d\frac{R}{\tau + 1} + q]}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{2dR(\tau + 1)^{-2}\frac{\partial \tau}{\partial c}q(1 + c) + q[2d\frac{R}{\tau + 1} + q]}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{2dR\frac{q(1+c)}{R-1+q}\frac{R-1+q}{q(1+c)^2}(-1)q(1 + c) + q[2d\frac{R}{R-1+q} + q]}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{2dR\frac{(1+c)}{R-1+q}(-1)q + q[2dR\frac{(1+c)}{R-1+q} + q]}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{q^2}{[2d\frac{R}{\tau + 1} + q]^2} > 0
\]

\[
\frac{\partial b}{\partial d} = \frac{\{-2\frac{R}{\tau + 1}\}[2d\frac{R}{\tau + 1} + q] - [cq - 2d\frac{R}{\tau + 1}]\frac{2R}{\tau + 1}}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{\{-2\frac{R}{\tau + 1}\}[2d\frac{R}{\tau + 1}] - \left\{\frac{2R}{\tau + 1}\right\}q - \left\{\frac{2R}{\tau + 1}\right\}cq + \left\{\frac{2R}{\tau + 1}\right\}\frac{2d\frac{R}{\tau + 1}}{[2d\frac{R}{\tau + 1} + q]^2}
= \frac{-\frac{2R}{\tau + 1}q(1 + c)}{[2d\frac{R}{\tau + 1} + q]^2} < 0
\]

\[
\frac{\partial b}{\partial q} = \frac{2(1 + c)^2dR}{(-1 + q + R + 2dR + 2cdR)^2} > 0
\]
\[
\frac{\partial b}{\partial R} = \left\{ -2d\frac{(\tau+1)-R^{2(\tau+1)}}{(\tau+1)^2} \right\} \left\{ 2d\frac{R}{\tau+1} + q \right\} - \left\{ cq - 2d\frac{R}{\tau+1} \right\} \left\{ 2d\frac{(\tau+1)-R^{2(\tau+1)}}{(\tau+1)^2} \right\} \]
\[
= \left\{ -2d\frac{R-1+q}{(\tau+1)^2} \right\} \left\{ 2d\frac{R}{\tau+1} + q \right\} - \left\{ cq - 2d\frac{R}{\tau+1} \right\} \left\{ 2d\frac{R-1+q}{(\tau+1)^2} \right\} \frac{R}{\tau+1} \]
\[
= \left\{ \frac{2d(1-q)}{q(1+c)(\tau+1)^2} \right\} \left\{ 2d\frac{R}{\tau+1} + q \right\} - \left\{ cq - 2d\frac{R}{\tau+1} \right\} \left\{ \frac{-2d(1-q)}{q(1+c)(\tau+1)^2} \right\} \frac{R}{\tau+1} \]
\[
= \left\{ \frac{2d(1-q)}{q(1+c)(\tau+1)^2} \right\} \frac{q(1+c)}{2d\frac{R}{\tau+1} + q} \left\{ 2d\frac{R}{\tau+1} + q \right\} - \left\{ cq - 2d\frac{R}{\tau+1} \right\} \left\{ \frac{2d(1-q)}{q(1+c)(\tau+1)^2} \right\} \frac{R}{\tau+1} \]
\[
= \left\{ \frac{2d(1-q)}{(\tau+1)^2} \right\} \frac{2d(1-q)}{2d\frac{R}{\tau+1} + q}^2 > 0
\]

\[
\square
\]

**Proposition 2.** The risky holdings in the competitive equilibrium \((n)\) are increasing in the return to the risky asset \((R)\), and decreasing in the size of the liquidity shock \((c)\), marginal cost of funds \((d)\), and the probability of the bad state \((q)\).

Proof.

\[
\frac{\partial n}{\partial c} = \frac{(-1 + c)q - 2(-1 + (1 + d + cd)R)}{2(1 + c)^3d} < 0 \text{, not obvious but clear after little algebra}
\]

\[
\frac{\partial n}{\partial d} = \frac{1 + cq - R}{2(1 + c)^2d^2} < 0 \text{ due to assumption } R - cq > 1
\]

\[
\frac{\partial n}{\partial R} = \frac{(-1 + q + R)^2 - 2(1 + c)d(cq^2 - (-1 + R)^2 - q(-1 + c + 2R))}{2(1 + c)^2d(-1 + q + R)^2} > 0
\]

\[
\left( -1 + q + R \right)^2 - 2(1 + c)d(cq^2 - (-1 + R)^2 - q(-1 + c + 2R)) = \text{ positive}
\]

\[
(-1 + q + R)^2 - 2(1 + c)d(cq (q - 1) - (-1 + R)^2 - q(-1 + 2R)) > 0
\]

\[
\text{negative positive}
\]

17
\[ \frac{\partial n}{\partial q} = -\frac{2d(-1 + R)R + 2c^2d(-1 + R) + c(q^2 + 2q(-1 + R) + (-1 + R)(-1 + R + 4dR)}{2(1 + c)^2d(-1 + q + R)^2} < 0 \]

Not obvious but at a closer look clearly negative.

To sum up; \( b \) and \( n \) move in the same direction as response to following parameters: \( R \) and \( d \), while they move in opposite directions as response to \( c \) and \( q \).

This is intuitive for \( cq \) is the expected value of liquidity need at the interim period. As the expected liquidity need increases, the bank holds more liquidity and less risky asset. Of course that does not say whether the bank increases enough its liquidity holdings, from a socially optimal perspective.

5 Partial Regulation: Regulating only capital ratios

In this section we assume that regulators constrain only leverage \( (n_i) \) but allow banks to freely choose their liquidity ratio \( (b_i) \). We consider this case to mimic the regulatory framework in the pre-Basel III period, which predominantly focused on capital adequacy requirements. Banks’ leverage ratio has to satisfy \( n_i \leq n \) where \( n \) is the maximum leverage level set by the regulator. We will start by assuming that banks leverage up to the allowed maximum level, i.e., banks’ choice of \( n_i \) is assumed to be equal to \( n \) that is determined by the regulator. Later, we will prove this assertion.

Given \( n_i = n \), the first order condition of banks w.r.t. \( b_i \)

\[ (1 - q) + qR \frac{1}{P} = D'(n(1 + b_i)) \implies b_i = \frac{D^{-1}(1 - q + qR \frac{R}{P})}{n} - 1 \]

Taking this into account, the social planner’s problem is

\[ \max_{n_i} \Pi_i(n_i) = (1 - q)\{R + b_i(n_i)\}n_i + qR\gamma_i n_i - D(n_i(1 + b_i(n_i))) \quad (17) \]

\[ (1 - q){R + b(n) + n b'(n)} + qR \{ \gamma_i + n \frac{\partial \gamma_i}{\partial n} \} = D'(n(1+b))\{1+b(n) + n b'(n) \} \]

Lemma 5. Equilibrium price of assets at \( t = 1 \) is decreasing in \( n \) and increasing in \( b \).

Proof. We can solve for the equilibrium price of assets at \( t = 1 \) by equating the demand and supply curves:

\[ (1 - \gamma)n = R \frac{P}{n} - 1 = \frac{c - b}{P}n \quad \implies \quad P = R + (b - c)n \]
It is clear from this equation that $P$ is decreasing in $n$ since $b - c < 0$ and increasing in $b$.  

**Proposition 3.** Banks decrease their liquidity ratio as the regulator tightens capital requirements, i.e. $b$ is decreasing in $n$.

**Proof.** Using the banks’ FOC above and plugging in the functional form for the cost function we can obtain $b$ as an implicit function of $n$ (note that $P$ is a function of both $b$ and $n$).

$$(1 - q) + qR\frac{1}{P} = 1 + 2dn(1 + b_i) \implies b_i = \frac{(1 - q + qR)}{2dn} - 1 = \frac{q(R - 1)}{2dn} - 1$$

From here we can obtain $\frac{\partial b}{\partial n} \equiv b'(n)$ as follows:

$$\frac{\partial b}{\partial n} \equiv b'(n) = \frac{-1}{P^2} \frac{\partial P}{\partial n} \frac{2dn}{4d^2n^2} - \frac{2dq(R - 1)}{P^4} \frac{4d^2n^2}{4d^2n^2}$$

From $P = R + [b(n) - c]n$ we have that $\frac{\partial P}{\partial n} = b(n) - c + b'(n)n$. Plug this into the equation above to get:

$$b'(n) = \frac{-2dq\frac{R}{P^2}[b(n) - c + b'(n)n]}{4d^2n^2} - \frac{2dq(R - 1)}{P^4}$$

and hence we can get

$$4d^2n^2b'(n) = -2dq\frac{nR}{P^2}[b(n) - c] - 2dq\frac{R}{P^2} b'(n)n - 2dq\frac{R}{P}$$

Solving this last equation for $b'(n)$ gives

$$b'(n) = \frac{-2dq\frac{nR}{P^2}[b(n) - c] - 2dq\frac{R(R - 1)}{P^2}}{4d^2n^2 + 2dn^2q\frac{R}{P^2}} > 0$$

The sign is positive because we can show that the sign of term inside the brackets in the numerator is negative. This is shown below:

$$\frac{nR}{P^2}(b - c) + \left(\frac{R}{P} - 1\right) = \frac{R(b - c)n + RP - P^2}{P^2} = \frac{R(b - c)n + P(R - P)}{P^2} = \frac{R(b - c)n + P(R - (b - c)n)}{P^2} = \frac{R(b - c)n + P(b - c)n}{P^2} = \frac{(R - P)(b - c)n}{P^2} < 0$$
because \( R - P > 0 \) and \( b - c < 0 \).

Now let’s look at what happens to \( \gamma \) as \( n \) increases. To see that we evaluate the total derivative below:

\[
\frac{d\gamma}{dn} = \frac{\partial\gamma}{\partial b} \frac{db}{dn} + \frac{\partial\gamma}{\partial P} \frac{dP}{dn}
\]

Compared to Kara (2013) the first term in the total derivative is extra. In Kara (2013) \( d\gamma/dn < 0 \) because \( n \) affects \( \gamma \) only through \( P \), which decreases as \( n \) increases. Here \( n \) affects \( \gamma \) through \( b \) as well. Increasing \( b \) increases gamma directly, i.e. \( \partial\gamma/\partial b > 0 \). Moreover, we have shown above that \( db/dn > 0 \), i.e. agents increase their liquidity ratio as the regulator relaxes the capital ratio (to allow more leverage). Therefore, the sign of the first (extra) term is clearly positive. But since the second term is negative, the sign of \( d\gamma/dn \) is cannot be determined from here. By some algebra we can show that this total derivative can be written as

\[
\frac{d\gamma}{dn} = \frac{b'(n)R - [b(n) - c]^2}{P^2}
\]  

(20)

### 5.1 Closed form solutions

Using the first order conditions of the social planner’s problem given by equation (17) we can obtain the closed form solutions for \( n \) and \( b \) in this case after lengthy calculus and algebra as follows:

\[
n^p = \frac{\left[-cq^2 + 8(1 + c)d^2R^2 + q\kappa\sqrt{\rho} - (cq^{3/2} + \sqrt{q}\kappa)\sqrt{\sigma}\right]}{4(1 + c)^2d\sqrt{p}(q + 2dR)}
\]  

(21)

\[
b^p = \frac{-q^2 + 8d^2R^2 + q(R - 1 + 4dR) - \sqrt{q}\rho\sigma}{4dR(q + 2dR)}
\]  

(22)

where

\[
\rho \equiv R - 1 + q
\]  

(23)

\[
\kappa \equiv -1 + [1 + 2(1 + c)d]R
\]  

(24)

\[
\sigma \equiv (q - 1)q + [1 + 8(1 + c)d]qR + 16(1 + c)d^2R^2
\]  

(25)

### 6 Complete Regulation: Regulating both capital and liquidity ratios

Assuming the constraint always binds, i.e. \( b < c \),
\[
\max_{n,b} \Pi_i(n, b) = (1 - q)(R + b)n + qR \gamma n - D(n(1 + b))
\] (26)

subject to the budget constraint at \( t = 1 \)

\[
P(1 - \gamma)n + bn - cn \geq 0
\]

Corresponding first order conditions with respect to \( n \) and \( b \) are respectively;

\[
(1 - q)(R + b) + qR \left\{ \gamma + (b - c)P^{-2}(1) \frac{\partial P}{\partial n} \right\} = D'(n(1 + b))(1 + b) \quad \text{where} \quad \gamma = 1 + \frac{b - c}{P}
\]

\[
(1 - q)n + qR \left\{ \frac{1}{P} - (c - b)P^{-2}(1) \frac{\partial P}{\partial b} \right\} n = D'(n(1 + b))n
\]

Further simplification yields

\[
(1 - q)(R + b) + qR \left\{ \gamma + (-1)(1)P^{-1} \frac{\partial P}{\partial n} \right\} = D'(n(1 + b))(1 + b) \quad \text{where} \quad \gamma = 1 + \frac{b - c}{P}
\]

extra and negative

\[
(1 - q)n + qR \left\{ \frac{1}{P} + (1 - \gamma)P^{-1} \frac{\partial P}{\partial b} \right\} n = D'(n(1 + b))n
\]

extra and positive

For comparison, the corresponding FOCs in the competitive equilibrium;

\[
(1 - q)(R + b) + qR \gamma = D'(n(1 + b))(1 + b) \quad \text{where} \quad \gamma = 1 + \frac{b - c}{P}
\]

\[
(1 - q)n + qR \frac{1}{P} n = D'(n(1 + b))n
\]

If we make functional assumptions; demand side \( y = \frac{R}{P} - 1 \) which implies \( P = R + (b - c)n \). So \( \frac{\partial P}{\partial n} = b - c \) and \( \frac{\partial P}{\partial b} = n \). Plugging in those;

\[
(1 - q)(R + b) + qR \left\{ \gamma + (-1)(1) \frac{2n}{P} \right\} = D'(n(1 + b))(1 + b) \quad \text{where} \quad \gamma = 1 + \frac{b - c}{P}
\]

extra and negative

\[
(1 - q)n + qR \left\{ \frac{1}{P} + \frac{(1 - \gamma)n}{P} \right\} n = D'(n(1 + b))n
\]

extra and positive
Given $P = R + (b - c)n$, we have two equations and two unknowns (given a functional form for $D(\cdot)$ as well). After some lengthy algebra we can obtain closed form solutions for both $n, b$ and $P$ as follows:

\[
\begin{align*}
n^s &= \frac{R - 1 + q}{1 + c} - q - 2d(P^s - R) \quad (27) \\
b^s &= \frac{2d(1 + c)(P^s - R)}{R - 1 + q - 2d(P^s - R)} + c \quad (28)
\end{align*}
\]

where

\[
P^s = \sqrt{qR^2(1 + c)} \quad (29)
\]

instead we can write these solutions as follows:

\[
\begin{align*}
n^s &= \frac{2dR\tau^s + q\tau^s(\tau^s + 1)(\tau^s + 2)}{2d(1 + c)(\tau^s + 1)} \quad (30) \\
b^s &= \frac{cq(\tau^s + 1)(\tau^s + 2) - 2dR}{2dR + q(\tau^s + 1)(\tau^s + 2)} \quad (31)
\end{align*}
\]

where

\[
\tau^s = \frac{R}{P^s} - 1 = \sqrt{\frac{R - 1 + q}{q(1 + c)} - 1} \quad (32)
\]
Solving for $b^\ast$.

Corresponding first order conditions with respect to $n$ and $b$ are respectively;

\[
(1 - q)(R + b) + qR\left\{ \gamma + \frac{\partial \gamma}{\partial n} n \right\} = D'(n(1 + b))(1 + b) \text{ where } \gamma = 1 + \frac{b - c}{P}
\]

\[
(1 - q)n + qR\frac{\partial \gamma}{\partial b} n = D'(n(1 + b))n
\]

Combining the two equations

\[
(1 - q)(R + b) + qR\left\{ \gamma + \frac{\partial \gamma}{\partial n} n \right\} = \left[ (1 - q) + qR\frac{\partial \gamma}{\partial b} \right] (1 + b) = D'(n(1 + b))(1 + b)
\]

Plugging $\frac{\partial \gamma}{\partial n} = -\frac{(b - c)^2}{P^2}$ and $\frac{\partial \gamma}{\partial b} = \frac{R}{P^2}$, and later $P = R + (b - c)n$.

\[
-\phi = \frac{(1 - q)[R - 1]}{qR} = 1 + \frac{(b - c)P - (b - c)^2 n - R(1 + b)}{P^2}
\]

\[
-\phi - 1 = \frac{(b - c)[R + (b - c)n] - (b - c)^2 n - R(1 + b)}{P^2} = \frac{R(b - c - 1 - b)}{P^2} = -\frac{R(c + 1)}{P^2}
\]

\[
\Rightarrow P^2 = \frac{R(c + 1)}{\phi + 1} = \frac{q(c + 1)R^2}{R - 1 + q} \Rightarrow P^{**} = R\sqrt{\frac{q(c + 1)}{R - 1 + q}}
\]

Note that $P = \frac{P^{**2}}{R}$.

Going back to the first order condition w.r.t. $b$
\[(1 - q) + qR \frac{\partial \gamma}{\partial b} = D'(n(1 + b))\]

\[1 - q + qR \frac{R}{P^2} = 1 + 2dn(1 + b)\]

\[1 - q + q \left( \frac{R}{P} \right)^2 = 1 + 2dn(1 + b)\]

\[1 - q + q(\tau^s + 1)^2 = 1 + 2dn(1 + b)\]

\[q \{(\tau^s + 1)^2 - 1\} = 2dn(1 + b)\]

\[q\{(\tau^s + 1)(\tau^s + 1 - 1) = 2d \frac{P\tau^s}{c - b}(1 + b)\]

\[q\tau^s(\tau^s + 2) = 2d \frac{R}{\tau^s + 1} \frac{\tau^s}{c - b}(1 + b)\]

\[q(\tau^s + 1)(\tau^s + 2)(c - b) = 2dR(1 + b)\]

\[q(\tau^s + 1)(\tau^s + 2)c - 2dR = b\{2dR + q(\tau^s + 1)(\tau^s + 2)\}\]

\[b^s = b^{**} = \frac{cq(\tau^s + 1)(\tau^s + 2) - 2dR}{2dR + q(\tau^s + 1)(\tau^s + 2)} = \frac{cq - \frac{2dR}{(\tau^s + 1)(\tau^s + 2)}}{\frac{2dR}{(\tau^s + 1)(\tau^s + 2)} + q}\]

we used \( \frac{R}{\tau^{**}} = \tau^{**} + 1 \) and \( n^{**} = \frac{P^{**}}{c - b} \).
Lemma 6. $b^{**} > b > b^*$

\[ b = \frac{cq - 2d \frac{R}{\tau + 1}}{2d \frac{R}{\tau + 1} + q} \]  
\[ b^{**} = \frac{cq - 2d \frac{R}{(\tau^s + 1)(\tau^s + 2)}}{2d \frac{R}{(\tau^s + 1)(\tau^s + 2)} + q} \]  

where

\[ \tau = \frac{R - 1 + q}{q(1 + c)} - 1 = \eta^2 - 1 \]  
\[ \tau^s = \frac{R}{P^s} - 1 = \sqrt{\frac{R - 1 + q}{q(1 + c)}} - 1 = \eta - 1 \]  

\[ (\tau^s + 1)(\tau^s + 2) = \eta(\eta + 1) > \tau + 1 = \eta^2 \implies b^{**} > b. \]

**Part 2: $b > b^*$**

Let’s denote $b - b^*$ as the difference of two fractions $\frac{A}{B} - \frac{C}{D}$ where the capital letters refer to the numerator and denominators of $b$ and $b^*$ respectively, and they are all positive. We would like to show that

\[ \frac{A}{B} - \frac{C}{D} = \frac{AD - BC}{BD} > 0. \]  

Since $BD > 0$, it will be sufficient to show that $\Delta \equiv AD - BC > 0$. We will progress by showing that $\Delta = 0$ when $d = 0$ and that $\frac{d\Delta}{dd} > 0$ which will imply that $\Delta > 0$ always holds.

Mathematica file `compare_b_1` gives $\Delta$ for $d = 0$ as follows:

\[ \Delta_{d=0} = q[q^2 + q(-1 + R) - \sqrt{q^2(q - 1 + R)^2}] \]
\[ = q[q(q - 1 + R) - q(q - 1 + R)] \]
\[ = 0 \]

From the Mathematica file we also have that:

\[ \frac{d\Delta}{dd} = -\frac{2(1 + c)qR(3q + 8dR)[\sqrt{q}(-1 + q + (1 + 4(1 + c)d)dR) - \sqrt{\rho\sigma}]}{\sqrt{\rho\sigma}} \]  

Let’s look at the term within brackets in the numerator of this derivative. The derivative is positive if this term is negative. Below we establish that this is the case.

\[ \sqrt{q}(-1 + q + (1 + 4(1 + c)d)dR) < \sqrt{\rho\sigma} \]  

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Square of both sides:
\[ q(-1 + q + R + 4(1 + c)dR)^2 < \rho\sigma \] (41)

\[ q(\rho + 4(1 + c)dR)^2 < \rho[\rho + 8(1 + c)dqR + 16(1 + c)d^2R^2] \] (42)

\[ q\rho^2 + 16(1 + c)^2d^2R^2q + 8(1 + c)dqR\rho < q\rho^2 + 8(1 + c)dqR\rho + 16(1 + c)d^2R^2 \rho \] (43)

Cancel the same terms on both sides to get:
\[ 16(1 + c)^2d^2R^2q < 16(1 + c)d^2R^2 \rho \] (44)

Further simplification yields:
\[ 16(1 + c)^2d^2R^2q < 16(1 + c)d^2R^2 \rho \] (45)

\[ (1 + c)q < \rho = R - 1 + q \] (46)

or
\[ 0 < R - 1 - qc \] (47)

which is true by Assumption \textit{Range}.

\textbf{Lemma 7.} \( n > n^{**} > n^* \)

We will use \( n^{**} = \frac{P^{**} \tau^{**}}{c-b} \) and \( c - b^{**} = \frac{2dR(1+b^{**})}{q(\tau^{**}+1)(\tau^{**}+2)} \) which we derived as we solve for \( b^{**} \).

\[ n^* = n^{**} = R \frac{\tau^s}{\tau^s + 1} \frac{q(\tau^s + 1)(\tau^s + 2)}{2dR(1 + b^{**})} = q \frac{\tau^s(\tau^s + 2)}{2d(1 + b^{**})} \]

With similar algebra for the competitive equilibrium we can derive the following \( n = \frac{q}{2d}(\frac{\tau^s}{1+b^{**}}) \).

\[ \frac{\tau^s(\tau^s + 2)}{1 + b^{**}} = \frac{(\eta - 1)(\eta + 1)}{1 + b^{**}} = \frac{\eta^2 - 1}{1 + b^{**}} < \frac{\eta^2 - 1}{1 + b} = \frac{\tau + 1}{1 + b} \text{ since } b^{**} > b \]

So, \( n > n^{**} \).

\textbf{Lemma 8.} \( 1 - \gamma > 1 - \gamma^* > 1 - \gamma^{**} \)

\textit{Proof.}
\[ 1 - \gamma = \frac{c - b}{P} \text{ together with } b^{**} > b^* \text{ and } P^{**} > P^* \implies 1 - \gamma^* > 1 - \gamma^{**} \]

For \( (1 - \gamma) - (1 - \gamma^*) \), Mathematica file \texttt{compare_firecomp_firepar} gives the following as numerator

\[ (1 - \gamma) - (1 - \gamma^*) \propto \Delta = 4(1 + c)d\sqrt{qR\rho}(q + 2dR) - q[\rho + 2(1 + c)dR]\{\sqrt{q}\rho + \sqrt{p}\sqrt{\sigma}} \]
$\Delta$ is equal to zero when $d = 0$. What we will show below is that $\frac{\partial \Delta}{\partial d} > 0$, i.e. $\Delta$ is increasing in $d$, therefore for positive values of $d$ $\Delta$ as well as the difference will be positive.

\[
\frac{\partial \Delta}{\partial d} = 2(1 + c)\sqrt{qR}\{q^2 + q(-1 + R) + 4dR\rho + 2\rho(q + 2dR) - A - B\}
\]
\[
= 2(1 + c)\sqrt{qR}\{q(R - 1 + q) + \rho(2q + 4dR + 4dR) - A - B\}
\]
\[
= 2(1 + c)\sqrt{qR}\{\rho(3q + 8dR) - A - B\}
\]

where $A = \frac{2\sqrt{q}\sqrt{\rho(3q + 8dR)}[\rho + 2dR(1 + c)]}{\sqrt{\sigma}}$ and $B = \sqrt{q}\sqrt{\rho}\sqrt{\sigma}$.

Multiplying by $\sqrt{\rho}$

\[
\sqrt{\sigma}\rho(3q + 8dR) - 2\sqrt{q}\sqrt{\rho(q + 4dR)[\rho + 2dR(1 + c)]} - \sqrt{q}\sqrt{\rho}\sigma > 0
\]

Thus we conclude that $\frac{\partial \Delta}{\partial d} > 0$, i.e. the difference we are after is increasing in $d$ and this together with the difference is equal to zero when $d$ is zero implies that the difference is positive for positive values of $d$.

So, $(1 - \gamma) > (1 - \gamma^*)$. \hfill $\Box$

**Lemma 9.** $(1 - \gamma)n > (1 - \gamma^*)n^* > (1 - \gamma**)n^{**}$

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Proof. This is about the total amount of fire sales. Given that the demand function for risky assets in the interim period is downward sloping (continuous and differentiable as well), the prices will be informative about the amount of fire sales.

\[(1 - \gamma)n = \tau = \frac{R}{P} - 1 \text{ and } P^{**} > P^* > P \implies (1 - \gamma^{**})n^{**} < (1 - \gamma^*)n^* < (1 - \gamma)n\]

\[\text{Lemma 10. } P^{**} > P^* > P\]

\[\text{Proof. Mathematica file compare_psoc_ppar gives } P_s - P_{par} = P^{**} - P^* \text{ as follows}\]

\[
P^{**} - P^* = q + \frac{4(1 + c)^3d^2qR^2}{(1 + c)^2dR \sqrt{(1 + c)q \rho}} - \sqrt{q \sqrt{\sigma}} \sqrt{\rho}
\]

\[
= q + \frac{4dR \sqrt{1 + c} \sqrt{q} - \sqrt{q \sqrt{\sigma}} \sqrt{\rho}}{(\sqrt{\rho})^2 - \sqrt{q \sqrt{\sigma}}}
\]

\[
= q \sqrt{\rho} + 4dR \sqrt{1 + c - \sqrt{\rho}}
\]

\[
= \frac{\sqrt{q} \sqrt{\rho} + 4dR \sqrt{1 + c - \sqrt{\rho}}}{\sqrt{\rho}}
\]

By proving \([\sqrt{q} \sqrt{\rho} + 4dR \sqrt{1 + c - \sqrt{\rho}}] > 0\) we will be done.

\[
[\sqrt{q} \sqrt{\rho} + 4dR \sqrt{1 + c}]^2 - \sigma = q \rho + 16(1 + c)d^2 R^2 + 8\sqrt{1 + cd}R \sqrt{q} \sqrt{\rho} - [qp + 8(1 + c) dq R + 16(1 + c)d^2 R^2]
\]

\[
= 8dR \sqrt{1 + c} \sqrt{q} [\sqrt{\rho} - \sqrt{1 + c} \sqrt{q}] > 0
\]

The last inequality is due to \(\rho = R - 1 + q > q(1 + c) = q + qc\) since \(R - 1 + cq > 0\).

Therefore, \(P^{**} > P^*\).

Part 2: Mathematica file compare_pcomp_ppar gives \(P_{par} - P_{comp} = P^* - P\) as follows
\[ P^* - P = -\frac{(1 + c)qR}{\rho} + \frac{\sqrt{q}(\sqrt{\bar{q}} - \sqrt{\bar{p}} + \sqrt{\bar{\sigma}})}{4d} \]
\[ = \frac{q - q^2 - (1 + 4(1 + c)d)qR + \sqrt{q}\sqrt{\bar{p}}\sqrt{\bar{\sigma}}}{4d\rho} \]
\[ = \frac{-q[R + q - 1] - 4(1 + c)dRq + \sqrt{q}\sqrt{\bar{p}}\sqrt{\bar{\sigma}}}{4d\rho} \]
\[ = \frac{-q[\rho + 4(1 + c)dR] + \sqrt{q}\sqrt{\bar{p}}\sqrt{\bar{\sigma}}}{4d\rho} \]
\[ > \frac{-q[\rho + 4(1 + c)dR] + \sqrt{q}\sqrt{\bar{q}(\rho + 4(1 + c)dR)}}{4d\rho} = 0 \]

For the last inequality we used \( \rho\sigma = q[\rho + 4(1 + c)dR]^2 + R - cq - 1 > q[\rho + 4(1 + c)dR]^2 \).

Therefore, \( P^* > P \).
References


